Math 441 Name____________________

Exam II solutions

Fall 2013 **<u>Show All Work</u>**

1) Find the determinant of

$$\begin{bmatrix} 2 & 1 & 3 & 4 & 2 & 2 \\ 4 & 0 & 2 & 0 & 3 & 0 \\ 6 & 0 & 1 & 0 & 1 & 0 \\ 2 & 3 & 2 & 3 & 5 & 0 \\ 1 & 0 & 3 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 & 2 & 0 \end{bmatrix}$$

$$= -2 \begin{vmatrix} 4 & 0 & 2 & 0 & 3 \\ 6 & 0 & 1 & 0 & 1 \\ 2 & 3 & 2 & 3 & 5 \\ 1 & 0 & 3 & 0 & 4 \\ 0 & 0 & 0 & 2 & 2 \end{vmatrix} = (-2)(-3) \cdot \begin{vmatrix} 4 & 2 & 0 & 3 \\ 6 & 1 & 0 & 1 \\ 1 & 3 & 0 & 4 \\ 0 & 0 & 2 & 2 \end{vmatrix} = (-2)(-3)(-2) \cdot \begin{vmatrix} 4 & 2 & 3 \\ 6 & 1 & 1 \\ 1 & 3 & 4 \end{vmatrix}$$

$$= -12 \cdot \begin{vmatrix} 0 & -10 & -13 \\ 0 & -17 & -23 \\ 1 & 3 & 4 \end{vmatrix} = -12(230 - 221) = (-12)(9) = -108$$

2) Let A be a matrix for which $A^2 = A$. Show that the determinant of A is either 0 or 1.

Since $A^2 = A$ we know $|A^2| = |A|$. This implies $|A|^2 = |A|$. Hence $|A|^2 - |A| = 0$. Factoring this equation give $|A|(|A| - 1) = 0$. Hence by the Zero Product Property $|A| = 0$ or $|A| = 1$.

3) Let **A** and **B** be 3x3 matrices where $|\mathbf{A}| = 4$, $|\mathbf{B}| = -2$, and $\mathbf{A}^t$ is the transpose of **A**. Find the following or state "can't be found" from the information given.

a) $|\mathbf{A}^{-1}| = 1/4$

b) $|A\mathbf{B}| = -8$

c) $|A + \mathbf{B}|$ Can't be found from the information given

d) $|\mathbf{B}^3| = -8$

e) $|5\mathbf{A}| = 125*4 = 500$

f) $|\mathbf{A}^t| = \mathbf{4}$

g) If **C** is the matrix that results when one row of matrix **A** (above) have been multiplied by ½ then

$|\mathbf{C}| = (1/2) \cdot |A| = 2$

4) True or false

a. The set of all 2x2 upper triangular matrices forms a vector space.
TRUE

b. A subset S = {v_1, v_2, v_3, …v_k} spans a vector space V if and only if the vector equation $c_1v_1 + c_2v_2 + c_3v_3 + \quad + c_kv_k = 0$ has only the trivial solution.
FALSE (This is the definition of linear independence not spanning)

5) Given

$$\begin{aligned} x - 2y + 1z &= -3 \\ 3x - 7y + 5z &= 2 \\ -2x + y + 8z &= -1 \end{aligned}$$

Cramers rule says z = ???? (Note: you do not have to solve)

$$z = \frac{\begin{vmatrix} 1 & -2 & -3 \\ 3 & -7 & 2 \\ -2 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & -2 & 1 \\ 3 & -7 & 5 \\ -2 & 1 & 8 \end{vmatrix}}$$

6) a) Find the adjoint of **A** = $\begin{bmatrix} 3 & 1 & 2 \\ 0 & 5 & 0 \\ 4 & 0 & 3 \end{bmatrix}$

Matrix of cofactors = $\begin{bmatrix} \begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 4 & 3 \end{vmatrix} & \begin{vmatrix} 0 & 5 \\ 4 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 4 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 5 & 0 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 0 & 5 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 15 & 0 & -20 \\ -3 & 1 & 4 \\ -10 & 0 & 15 \end{bmatrix}$

Adjoint = $\begin{bmatrix} 15 & -3 & -10 \\ 0 & 1 & 0 \\ -20 & 4 & 15 \end{bmatrix}$

b) USE the ADJOINT above to find the inverse of A.

det (A) = $5 \cdot \begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix} = 5$

Adj(A) = $\frac{1}{|A|} \cdot Adj(A) = \frac{1}{5}\begin{bmatrix} 15 & -3 & -10 \\ 0 & 1 & 0 \\ -20 & 4 & 15 \end{bmatrix} = \begin{bmatrix} 3 & -3/5 & -2 \\ 0 & 1/5 & 0 \\ -4 & 4/5 & 3 \end{bmatrix}$

7) Let V be a set on which two operations (addition and scalar multiplication) are defined. What 10 axioms must be satisfied for every u, v, and w in V, and every scalar c and d, before we can call V a vector space?

i. $\mathbf{u} + \mathbf{v} \in V$
ii. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
iii. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
iv. There is an element $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
v. There is an element $-\mathbf{u}$ such that $\mathbf{u} + -\mathbf{u} = -\mathbf{u} + \mathbf{u} = \mathbf{0}$.
vi. $c\mathbf{u} \in V$
vii. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
viii. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
ix. $(cd)\mathbf{u} = c(d\mathbf{u})$
x. $1 \cdot \mathbf{u} = \mathbf{u}$

8) If possible write $\mathbf{v} = (4, 5, 1)$ as a linear combination of $\mathbf{u_1} = (3, 5, -4)$, $\mathbf{u_2} = (2, 1, -5)$, and $\mathbf{u_3} = (-2, 1, 3)$.

$$\left[\begin{array}{ccc|c} 3 & 2 & -2 & 4 \\ 5 & 1 & 1 & 5 \\ -4 & -5 & 3 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} -1 & -3 & 1 & 5 \\ 5 & 1 & 1 & 5 \\ -4 & -5 & 3 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -1 & -5 \\ 5 & 1 & 1 & 5 \\ -4 & -5 & 3 & 1 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & -5 \\ 0 & -14 & 6 & 30 \\ 0 & 7 & -1 & -19 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -1 & -5 \\ 0 & 0 & 4 & -8 \\ 0 & 7 & -1 & -19 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -1 & -5 \\ 0 & 7 & -1 & -19 \\ 0 & 0 & 4 & -8 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & -5 \\ 0 & 7 & -1 & -19 \\ 0 & 0 & 1 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 0 & -7 \\ 0 & 7 & 0 & -21 \\ 0 & 0 & 1 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 0 & -7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Thus $\mathbf{v} = 2\mathbf{u_1} - 3\mathbf{u_2} - 2\mathbf{u_3}$

9) Why does $\mathbf{R}^2$ under the operations defined below fail to be a vector space? Identify at least one axiom that fails.

Addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, 0)$
Scalar Multiplication: $c(x, y) = (cy, cx)$

There is no zero element (3, 4) + (?, ?) = (3, 4). No matter what you use the second component of the result would be zero.

(It fails some other rules, I just mentioned one)

10) a) Let W = {(0,y): y $\in\mathbb{R}$} with the standard operations in R^2. Does W form a subspace of R^2? Why or why not? SHOW WORK.

Clearly W is a subset of $\mathbb{R}^2$. Consider $(0, y_1)$ and $(0, y_2)$ two elements of W.
Then $(0, y_1) + (0, y_2) = (0, y_1+y_2)$. Since $y_1 + y_2 \in\mathbb{R}$ we know that this lies in W. Thus W is closed under addition.

Now let c be any scalar (real number)
By definition $c(0, y_1) = (0, cy_1)$. But $cy_1 \in\mathbb{R}$, therefore (0, cy1) $\in$ W. Thus W is closed under scalar multiplication.

Since W is closed under addition and scalar multiplication we may conclude that W is a subspace of $\mathbb{R}^2$

11) Does the set W = {(x, y, 2) | x$\in\mathbb{R}$ and y $\in\mathbb{R}$} form a subspace of $\mathbb{R}^3$? Why or why not? Show work.

No. It is not closed under addition. $(x_1, y_1, 2) + (x_2, y_2, 2) = (x_1+x_2, y_1+y_2, 4)$. The last coordinate is not 2, therefore W is not closed under addition.

12) Does S = {(1, 1, 2), (1, 4, 5), (1, 7, 8)} span $\mathbb{R}^3$? Give a reason for your answer

To determine whether a(1,1,2) + b(1, 4, 5) + c(1,7, 8) = (x, y, z) has a solution for every possible (x, y, z) we solve the system

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 1 & 4 & 7 & y \\ 2 & 5 & 8 & z \end{array}\right]$$

This system has a solution for all possible (x, y, z) if and only if $|A| \neq 0$.

But $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 4 & 7 \\ 2 & 5 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 3 & 6 \end{vmatrix} = 18 - 18 = 0.$ Therefore S does NOT span $\mathbb{R}^3$.

13) Is S = {x^2, 3x+2, x^2 + 4} is linearly independent in P_2. Give a reason for your answer.

To determine whether $ax^2 + b(3x+2) + c(x^2+4) = 0$ has only (a,b,c) = (0,0,0) for a solution we solve the system

$$\left[\begin{array}{ccc|c} 0 & 2 & 4 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

This system has only the trivial solution if and only if $|A| \neq 0$.

But $|A| = \begin{vmatrix} 0 & 2 & 4 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 3 & 0 \end{vmatrix} = 0 - 12 = -12.$ Therefore S is linearly independent.